Joaquin @ Princeton University » MMP Learning Seminar

Minimal Model Program Learning Seminar 01/20/2021

Friday, January 29, 2021 5:54 PM

The Minimal Model Program:

Classify smooth proj complex varieties $X \subseteq \mathbb{P}^N$.

T_X the tangent bundle and Ω_X the cotangent bundle.

$\omega_X = \Omega_X^n$ is called the canonical line bundle.

$\omega_X \simeq \mathcal{O}_X(K_X)$ for some Cartier divisor K_X.

Aim: Understand the geometry of X using numerical properties of K_X.

$C \subseteq X$ curve, $\mathcal{L}$ a line bundle on X. $\mathcal{L}.C = \deg_C(i^*\mathcal{L})$, $i: C \hookrightarrow X$.

K_X is ample (resp. anti ample) if $K_X.C > 0$ (resp. < 0) for all $C \subseteq X$

K_X is numerically trivial if $K_X.C = 0$ for all curve $C \subseteq X$.

Definition:

We say that X is $\begin{cases} \text{Fano} \\ \text{Calabi-Yau} \\ \text{canonically polarized} \end{cases}$ if K_X is $\begin{cases} \text{antiample} \\ \text{numerically trivial} \\ \text{ample.} \end{cases}$

Example: If C is a Fano curve, then $C \simeq \mathbb{P}^1$.

If C is a CY curve, then C is an elliptic

If C is canonically polarized curve, C is a curve of genus ≥ 2.

Example: $X \subseteq \mathbb{P}^N$ a smooth hypersurface of degree d.

By adjunction,
$$\begin{aligned} K_X &\sim (K_{\mathbb{P}^N} + X)|_X \\ &\sim ((-N-1)H + dH)|_X \\ &\sim (d-N-1)H|_X \end{aligned}$$

$C \subseteq \mathbb{P}^2$.

If $d \leq N$, then X is Fano, quadric or a line,

If $d = N+1$, then X is CY, elliptic,

If $d \geq N+2$, then X is canonically polarized curve of gen type.

Obs: $E \times \mathbb{P}^1$ have $K_X . C = 0$ for some curves and $K_X . C < 0$ for others.

	Fano	CY	canonically polarized.
π_1	$\{0\}$	?	in general ∞.
Aut	linear alg groups.	?	finite groups
Bir	monstrous groups $Bir(\mathbb{P}^3)$	?	finite groups.
Geom	simple geom. (spheres)	?	complicated rich
Arithmetics	A lot of $\mathbb{Q}$-points	?	$\mathbb{Q}$-points in a proper Zariski closed.

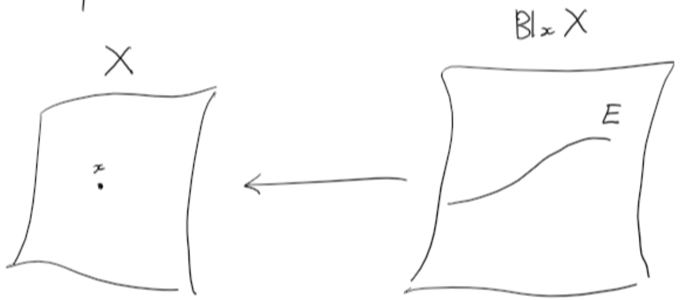
pure Calabi Yau's. $h^{i,0} = 0$, $i \in (0, \dim(X))$.
irred symplectic
complex tori.

Birational categories:

irred symplectic
complex. tori.

Birational categories:

x closed pt on X



E parametrizes tangent directions at x.

$$\mathrm{Bl}_x X \setminus E \simeq X \setminus \{x\}$$

Example: $\mathbb{P}^2$. $p_1, \dots, p_n, \dots \in \mathbb{P}^2$ "random" sequence of points.

blow-up these points in a sequence. We obtain a sequence of varieties

$X_1, X_2, \dots, X_n, \dots$ over $\mathbb{P}^2 \setminus \{p_1, \dots, p_{i-1}\}$, the morphism $X_i \longrightarrow \mathbb{P}^2$ is an isom.

For $i \neq j$. X_i is not isomorphic to X_j, because $\rho(X_i) \neq \rho(X_j)$.

Picard rank.

$X_1 \sim_{bir} X_2$ if there are dense opens $U_1 \subseteq X_1$, $U_2 \subseteq X_2$ so that $U_1 \simeq U_2$.

Goal of the MMP: X projective + "mild singularities". (K_X well-defined).

There exists a birational map π and a fibration φ such that

$$\begin{array}{c} X \overset{\pi}{\dashrightarrow} X' \\ \downarrow \varphi \\ Z. \end{array}$$

F will denote a general fiber of φ.

contraction + pos dim gen fiber

$\varphi_* \mathcal{O}_{X'} = \mathcal{O}_Z$.

(Connected fibers)

such that one of the following holds:

$\dim Z < \dim X'$

MFS $\longrightarrow$ i) F is Fano,

$\longrightarrow$ ii) F is Calabi-Yau,

$\longrightarrow$ iii) $Z = \operatorname{Spec}(\mathbb{C})$ and X' is canonically polarized.

How to construct the birational morphism?

Study the geometry of curves on X which intersect K_X negatively.

$C \subseteq X$, $K_X \cdot C < 0$ under some hypothesis (extremality on $NE(X)$).

we can find $\varphi_C : X \longrightarrow X_1$ so that φ_C contracts precisely the curves which are numerically equiv to qC, $q>0$

$C_1 \equiv C_2$.

$C_1 \cdot \mathcal{L} = C_2 \cdot \mathcal{L}$ for every l.b.

i) The curves which are $\equiv$ to qC, with $q>0$, cover X.

$\varphi_C : X \longrightarrow X_1$ have positive dim fibers, is a contraction, and the general fiber F is Fano.

called a Mori fiber space.

ii) The curves which are $\equiv$ to $\mathbb{R}C$ cover a divisor on X.

inverse of a blow-up of a smooth subvariety

In this case, we say that $\varphi_C : X \longrightarrow X_1$ is a divisorial contraction.

$\boxed{\rho(X_1) = \rho(X) - 1.}$ X_1 still has nice sing., so we iterate the process.

iii) Small contraction: The curves which are $\equiv$ to $\mathbb{R}C$ cover a set of codim ≥ 2.

X_1 has very bad singularities (K_{X_1} is not $\mathbb{Q}$-Cartier).

Construct a new birational morphism $\varphi_C^+ : X^+ \longrightarrow X_1$ which contracts K_{X^+}-positive curves.

Existence of flips.

Conjecture: Do flips always exist?

flip

$$\begin{tikzcd}
X \arrow[rr, dashed, "\pi"] \arrow[dr] & & X^+ \arrow[dl] \\
& X_1 &
\end{tikzcd}$$

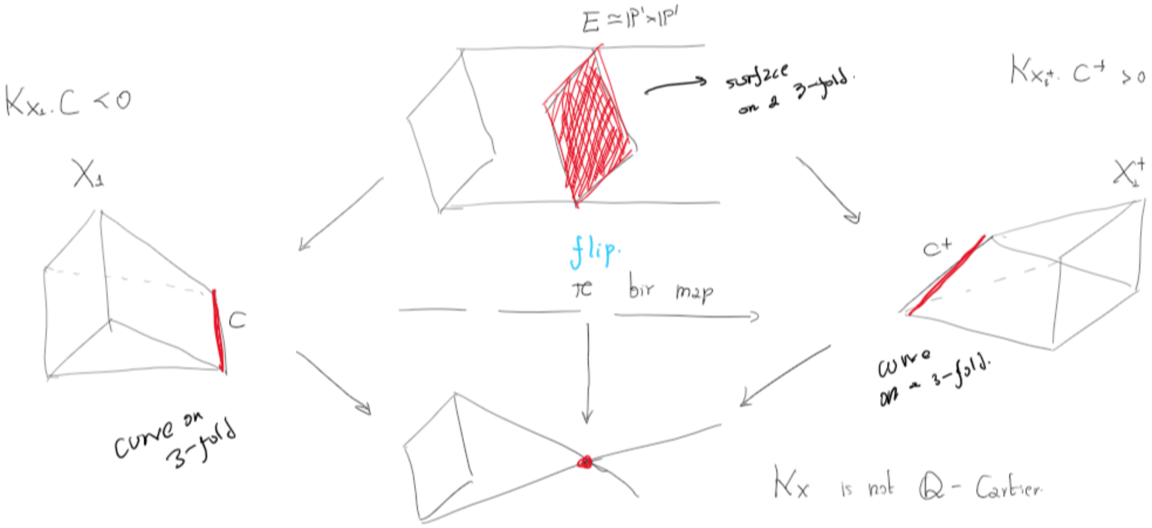
deleting K_X-neg curves and introducing K_{X^+}-pos curves

Flip is a small surgery (it only changes a loci codim ≥ 2).

$\boxed{\rho(X) = \rho(X^+).}$

Example: $\mathbb{P}^1\times\mathbb{P}^1$, $D = p_1^*\mathcal{O}(1)\otimes p_2^*\mathcal{O}(r)$ $\quad r>1$

$$X = \mathrm{Spec}\Big(\bigoplus_{m\geq 0} H^0(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(mD))\Big).$$



$$\begin{tikzcd} X \arrow[r, "dc"] & X_1 \arrow[r, "dc"] & X_2 \arrow[r, dashed, "flip"] & X_3 \arrow[r, dashed, "flip"] & X_4 \arrow[r, dashed] & \cdots \arrow[r, dashed] & X_n \arrow[d, "MFS"] \\ & & & & & & Z \end{tikzcd}$$

✓

$\rho(X)$.

Conjecture: Termination of flips. (every sequence of flips is finite).

nef

$K_{X_n} \cdot C \geq 0$ for every curve C. (K_{X_n} is numerically effective).

Conjecture (Abundance): X has mild sing and K_X is nef.

$|mK_X|$ is base point free for some $m \gg 0$.

$$X \xrightarrow{\varphi} X_1$$ contracts all K_X-trivial curves.

i) dim general fiber is > 0, in such case $K_F \equiv 0$.

ii) dim general fiber is $= 0$, $X \longrightarrow X_1$ is birational and X_1 is canonically pol.

Goal of the MMP is achieved if we can solve:

i) Existence of flips (we can run the MMP) ← BCHM 06

ii) Termination of flips (the MMP stops) known in dim ≤ 3 (some cases in dim 4)

iii) Abundance (When it stops, we have some nice fibration). known in dim ≤ 3

Structure of the seminar:

Part I:
- Kollár - Mori: Bir geom of alg var. (*)
- Kollár: Sing of the MMP.
- Lazarsfeld: Positivity in alg geometry I & II.

Feb - March

Part II:
- BCHM06. (*)
- Hacon - Kovacs: Higher dim alg varieties
- Several papers.

Apr - May - June.

Part III: Selected topics. (Over the summer).